Change of Variables for Double Integrals

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R region in the plane — under the graph of $z=f(x,y)$

— if R is a rectangle w/ sides parallel to coord. axes (i.e. $[a,b]\times[c,d]$), then we talked about to compute this

— if R has two horizontal sides (but other sides might not be)

Principle

If $R = R_1 \cup R_2$ and R_1 & R_2 don't overlap other than their boundary, then

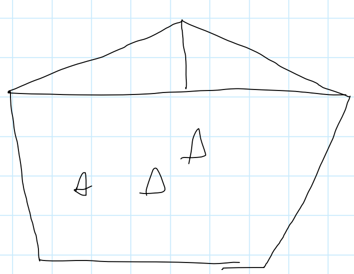
$$\iint_R f(x,y)\,dx\,dy = \int$$

Key: one side of R is parallel to one of the coordinate axis.

More generally: For any triangle, you have two options:

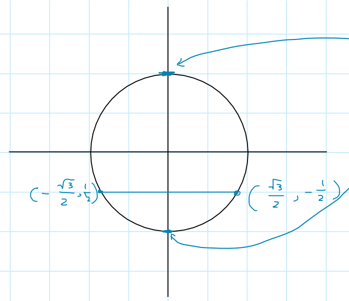
1. rotate so that one side's parallel (technically uses change of variables)
2. break up any triangle into pieces that have one side parallel to one of the coordinate axes

can do something similar if R is a polygon



Q/ What about integrating over the circle $R = \{(x,y)\in \mathbb{R}^2 \mid x^2+y^2 \le 1\}$

A/ can pretend R has 2 horizontal sides



horizontal sides

top horizontal side: $y = 1$

bottom: $y = -1$

Fix y, say $y = -\frac{1}{2}$

Notice

$$\frac{\sqrt{3}}{2} = \sqrt{1 - \left(-\frac{1}{2}\right)^2}$$

For each y btwn -1 & 1 x goes from $-\sqrt{1-y^2}$ to $\sqrt{1-y^2}$

so

$$\iint_R f(x,y)\,dx\,dy = \int_{y=-1}^{y=1} \int_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

↳ This used ②

↳ can also use ③. Then your vertical lines are $x = -1$ and $x = 1$ then you get:

$$\int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} f\,dy\,dx$$

e.g. $F(x,y) = 1$ (const fcn)

Recall $\iint_R 1\,dx\,dy = \text{area}(R)$

Try this for R = unit disc.

$$\int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} 1\,dy\,dx$$

$$= \int_{x=-1}^{x=1} y \Big]_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}$$

$$= \int_{x=-1}^{x=1} \left(\sqrt{1-x^2}\right) - \left(-\sqrt{1-x^2}\right) dx$$

$$= \int_{-1}^{1} 2\sqrt{1-x^2}\,dx$$

↖ This is the integral we know from single-var calc for the area of a circle.

↖ can use Trig Sub to evaluate

Trigonometric Substitution uses change of variables formula in calculus can use Trig Sub to evaluate

Trignometric Substitution uses change of variables formula in calculus.

Review

$x = \cos u$

$$\int_{x=-1}^{x=1} 2\sqrt{1-x^2}\,dx = \int_{x=-1}^{x=1} 2|\sin u|\,dx$$

Change of limits of integration

$x=-1 \quad \cos(u) = -1 \Rightarrow u = \pi$

$x = 1 \quad \cos(u) = 1 \Rightarrow u = 2\pi$

$$\int_{u=\pi}^{u=2\pi} 2|\sin u|\,dx \longleftarrow$$

Note
- for $\pi \leq u \leq 2\pi$
  $\sin u \leq 0$
- therefore
  $|\sin u| = -\sin u$

$$= \int_{u=\pi}^{u=2\pi} -2\sin u\,dx$$

$$= \int_{\pi}^{2\pi} 2\sin^2 u\,du$$

Key: fund thm of calc req taking antiderivative wrt variable inside the d.

need $dx \to du$

How?

$dx = \frac{dx}{du} \cdot du$

$= -\sin u\,du$

Key idea

$dx = \frac{dx}{du}\,du$

Q/ Can we find $\iint_R dx\,dy$ using polar coords? (w/ R = unit disc.)

Q/ why is this helpful?

A/ Unit disc R has simple description in polar coords (r,θ):

$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

this is really just the rectangle $[0,1] \times [0,2\pi]$ in (r,θ) coords

↳ so this reduces to 1

$$\iint 1\,dx\,dy = \iint 1\,dx\,dy$$

R $(r,\theta) \in [0,1]\times[0,2\pi]$

Q/ How to convert btwn $dx\,dy$ & $dr\,d\theta$?

ie $dx\,dy = [\text{what}]\,dr\,d\theta$

A/ say we want to diff (x,y) wrt (r,θ)

↳ that's what the [what should be]

Q/ what is x,y in terms of r,θ?

A/ $x = r\cos\theta$

$y = r\sin\theta$

↳ this is really a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

↳ its deriv. is a 2×2 matrix

$$\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

attempted ans:

$$\frac{dx}{dy} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} dr\,d\theta$$

$$\Rightarrow \iint dx\,dy = \iint_{[0,1]\times[0,2\pi]} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} dr\,d\theta$$

⚠ PROBLEM ⚠

↳ this is a matrix, not a scalar & A/V are scalar

Q/ How to turn a matrix into a scalar?

A/ determinant!

$$\frac{dx}{dy} = \det\left(\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}\right) dr\,d\theta$$

in this case:

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$\boxed{dx\,dy = r\,dr\,d\theta}$$

$$\Rightarrow \iint 1\,dx\,dy = \int^{\theta=2\pi}\int^{r=1} r\,dr\,d\theta$$

$$\Rightarrow \iint_R 1\,dx\,dy = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} r\,dr\,d\theta$$

Using method ①

$$\int_{\theta=0}^{\theta=2\pi} \left[\frac{r^2}{2}\right]_{r=0}^{r=1} d\theta = \int_0^{2\pi} \frac{1}{2}\,d\theta = 2\pi\left(\frac{1}{2}\right) = \pi$$

Q/ What does $\iint_R f(x,y)\,dx\,dy$ mean?

Recall

$\int_{[a,b]} f(x)\,dx$ is essentially just the value of f times the length (or change in x) of the interval.

Caveat f doesn't necessarily take one single value of the whole interval

Solution Riemann's sums!

↳ break [a,b] into little pieces on which f doesn't vary too much

↳ so, you can think of f as having approx constant value on each interval

↳ then make more & more intervals, w/ smaller & smaller intervals and take the limit as the mesh goes to 0

↳ max len of an interval among the intervals you broke [a,b] into

- Usually just use intervals that each have length $\frac{b-a}{N}$ $\Rightarrow$ mesh $= \frac{b-a}{n}$

↳ now say